Adding across the cyclic permutations of this last inequality gives the second desired result.

The overall inequality follows from the two results we have obtained.

Also solved by Arkady Alt, San Jose, CA, USA; and the proposer.

3350. [2008 : 241, 243] *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let x, y, and z be positive real numbers such that $x+y+z=1$. Prove that

$$\frac{yz}{1+x}+\frac{zx}{1+y}+\frac{xy}{1+z} \;\leq\; \frac{1}{4}\,.$$

I. *Similar solutions by George Apostolopoulos, Messolonghi, Greece; Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; Khanh Bao Nguyen, High School for Gifted Students, Hanoi University of Education, Hanoi, Vietnam; Babis Stergiou, Chalkida, Greece; Son Hong Ta, Hanoi, Vietnam; and Titu Zvonaru, Comănești, Romania.*

For positive real numbers a, b, and c we have that $(b+c)^2 \geq 4bc$, hence $\frac{a}{b+c} \leq \frac{a}{4}\left(\frac{1}{c}+\frac{1}{b}\right)$. Thus,

$$\begin{aligned}\sum_{\text{cyclic}}\frac{yz}{1+x} &= \sum_{\text{cyclic}}\frac{yz}{(x+y)+(z+x)} \;\leq\; \sum_{\text{cyclic}}\frac{yz}{4}\left(\frac{1}{x+y}+\frac{1}{z+x}\right)\\ &= \sum_{\text{cyclic}}\frac{xy+zx}{4(y+z)} \;=\; \sum_{\text{cyclic}}\frac{x}{4}=\frac{1}{4}\,.\end{aligned}$$

Equality holds if and only if $x=y=z=\frac{1}{3}$.

II. *Solution by Arkady Alt, San Jose, CA, USA, condensed by the editor.*

Let $e_1=x+y+z$, $e_2=xy+yz+zx$, $e_3=xyz$, and $S=\sum'\frac{xy}{kz+1}$ ($\sum'$ denotes a cyclic sum over x, y, z). We will prove that if k, x, y, z are positive real numbers and $e_1=1$, then $S\leq\max\left\{\frac{1}{4},\,\frac{1}{k+3}\right\}$.

Let $k\in(0,1]$. Since $S=e_3\sum'\left(\frac{1}{z}-\frac{k}{kz+1}\right)=e_2-ke_3\sum'\frac{1}{kz+1}$ and $\sum'\frac{1}{kz+1}\geq 9\left(\sum'(kz+1)\right)^{-1}=\frac{9}{k+3}$, it follows that $S\leq e_2-\frac{9ke_3}{k+3}$. It therefore suffices to prove that $e_2-\frac{9ke_3}{k+3}\leq\frac{1}{k+3}$, which is equivalent to

$$(k+3)e_2-9ke_3 \;\leq\; 1\,. \tag{1}$$

The inequality (1) follows from the two inequalities

$$e_2 \;\geq\; 9e_3\,, \tag{2}$$
$$4e_2 \;\leq\; 1+9e_3\,, \tag{3}$$

since $1-(k+3)e_2+9ke_3 = (1-4e_2+9e_3)+(1-k)(e_2-9e_3)$ and $k \leq 1$. Now, (2) follows from $3\sqrt[3]{e_3} \leq e_1 = 1$ and $3\sqrt[3]{e_3^2} \leq e_2$, and these follow from the AM–GM Inequality, while (3) follows by rewriting the Schur Inequality modulo the relation $e_1 = 1$; that is, one rewrites $\sum' x(x-y)(x-z) \geq 0$ as $2e_1^3 - 6e_1e_2 - e_1^2 + 2e_2 + 9e_3 \geq 0$ and puts $e_1 = 1$. This completes the proof of the inequality for $k \in (0,1]$.

Note that if $k = 1$, then we obtain the original inequality to be proved, while if $k > 1$ then $S = S(k) < S(1) \leq \frac{1}{4} = \max\left\{\frac{1}{4}, \frac{1}{k+3}\right\}$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; REBECCA EVERDING and JENNIFER PAJDA, students, Southeast Missouri State University, Cape Girardeau, MO, USA; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; WEI-DONG, Weihai Vocational College, Weihai, Shandong Province, China; NGUYEN THANH LIEM, Tran Hung Dao High School, Phan Thiet, Vietnam; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; D.P. MEHENDALE (Dept. of Electronics) and M.R. MODAK, (formerly of Dept. Mathematics), S. P. College, Pune, India; TRAN THANH NAM, Tomsk Polytechnic University, Tomsk, Russia; STAN WAGON, Macalester College, St. Paul, MN, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

The following solvers submitted multiple solutions: Alt (5 solutions)*, Apostolopoulos* (3 solutions) *and Cao* (2 solutions).

Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina indicated that since $x+y+z=1$, our problem appears as problem 35 (solved on pp. 48-49) in the book Old and New Inequalities *by T. Andreescu, V. Cîrtoaje, G. Dospinescu, and M. Lascu; GIL Publishing House.*